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Almost sure equidistribution in expansive families ☆

by Michael Björklund and Daniel Schnellmann

Royal Institute of Technology (KTH), Department of Mathematics, S-100 44 Stockholm, Sweden

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ABSTRACT

In this paper we study generic equidistribution in families of sequences of points on tori. We assume that the sequences are parameterized by some subset of a Euclidean space, and we formulate geometric conditions on the dependence so that equidistribution holds almost everywhere with respect to the Lebesgue measure on the parameter space. As a consequence, we can give a new proof of an old result by Koksma.

1. INTRODUCTION

Equidistribution of sequences of real numbers modulo 1 is a very classical and well-studied area of research (see e.g. [2,3,5] and the references therein). The main inspiration for this paper is the following classical and well-known result by Koksma [4] which can also be generalized to a larger class of real sequences: For almost every $\theta > 1$, with respect to the Lebesgue measure on $\mathbb{R}$, the sequence $\theta^j \bmod 1$, $j \geqslant 1$, is equidistributed in $\mathbb{T}$, i.e. for any interval A of the unit circle $\mathbb{T}$,

$$\frac{\#\{j;\theta^j \bmod 1 \in A, 1 \leqslant j \leqslant N\}}{N} \to |A|, \quad \text{as } N \to \infty,$$

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E-mails: mickebj@math.kth.se (M. Björklund), dansch@math.kth.se (D. Schnellmann).

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Koksma proved this result using Fourier analysis and reduced the statement to the combinatorics of exponential sums.

In this paper we will discuss a geometric method to prove Koksma's result based on the techniques developed by Benedicks and Carleson [1] in one-dimensional dynamics. This method can also be used to establish higher dimensional analogues of Koksma's theorem.

Let I be an open set in $\mathbb{R}^d$, and let $\tilde{f}_j$ be a sequence of maps on I into $\mathbb{R}^d$. Define, for a fixed θ in I, the sequence of points $f_j(\theta) = \tilde{f}_j(\theta) \bmod \mathbb{Z}^d$ in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

What are natural geometric conditions on the sequence $\tilde{f}_j$ so that we can ensure equidistribution of the sequence $f_j(\theta)$ in $\mathbb{T}^d$, for Lebesgue almost every θ in I?

Koksma studied the case $\tilde{f}_j(\theta) = \theta^j$, and $I = (1, \infty) \subset \mathbb{R}$. Two special features of this example are expansion and distortion. Here, expansion refers to the fact that

$$\frac{\theta^{j+k}}{\theta^j} = \theta^k$$

is growing sufficiently fast in k, for all $j \geqslant 1$, and distortion simply means that the quotient of the derivatives of $\tilde{f}_j$, restricted to the pre-image of a unit interval in $(1, \infty)$ is close to 1, provided that j is sufficiently large. We will see that these two properties are sufficient to conclude that, for almost every θ in I, the sequence $f_j(\theta)$ is equidistributed in $\mathbb{T}$. It is straightforward to generalize these two conditions to higher dimensions.

2. MAIN STATEMENT

Let

$$\begin{aligned} \tilde{f} &: \mathbb{N} \times I \to \mathbb{R}^d \\ (j, \theta) &\mapsto \tilde{f}_j(\theta), \end{aligned}$$

where $I \subset \mathbb{R}^d$ open and $d \geqslant 1$. For each $j \geqslant 1$ we assume that $\tilde{f}_j$ is a C^1 function in θ which is one-to-one and whose Jacobian is never 0. We put two conditions on $\tilde{f}$, one concerning the expansion and one concerning the distortion properties of $\tilde{f}$.

(I) There is $0 < \kappa < 1$ and an at least polynomially growing function $g : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$, i.e.

$$\varliminf_{n \to \infty} \frac{g(n)}{n^\gamma} > 0, \quad \text{for some } \gamma > 0,$$

such that for $1 \leqslant j \leqslant n$ and $k \geqslant n^\kappa$

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant g(n),$$

for all $\theta \in I$ and $v \in \mathbb{R}^d \setminus \{0\}$.

(II) For each $\varepsilon > 0$ and $r > 0$ there is an integer $j_{\varepsilon,r} \geqslant 1$ such that the following holds. Let $B(x,r)$ denote the open ball in $\mathbb{R}^d$ with radius r and center x. For all $x \in \mathbb{R}^d$ and all $\theta, \theta' \in \tilde{f}_j^{-1}(B(x,r) \cap \tilde{f}_j(I))$, $j \geqslant j_{\varepsilon,r}$, we have

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} \leqslant 1 + \varepsilon,$$

where $|D_\theta \tilde{f}_j(\theta)|$ is the Jacobian of $D_\theta \tilde{f}_j(\theta)$.

Remark. A weaker version of the first condition is:

(I)′ There is an at least polynomially growing function $g : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$, such that for $j \geqslant 1$ and $k \geqslant 1$,

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant g(k),$$

for all $\theta \in I$ and $v \in \mathbb{R}^d \setminus \{0\}$.

Obviously condition (I)′ implies condition (I) (with an arbitrarily chosen κ). The main reason for stating a refined version in condition (I) is that we want to include examples as $\tilde{f}_j(\theta) = \theta^{\sqrt{j}}$ (see Example 4.1). The introduction of the constant κ in (I) is natural in view of the estimate of the exceptional terms in the sum in (2).

Let Γ be a lattice in $\mathbb{R}^d$. We define $f : \mathbb{N} \times I \to \mathbb{R}^d/\Gamma$ as

$$f_j(\theta) = \tilde{f}_j(\theta) \bmod \Gamma.$$

Theorem 2.1. *If $\tilde{f}$ fulfills conditions* (I) *and* (II), *then the sequence $f_j(\theta)$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for Lebesgue almost every $\theta \in I$, i.e.*

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{f_j(\theta)} \xrightarrow{\text{weak-}*} m, \quad \text{as } n \to \infty,$$

where m denotes the Haar measure on $\mathbb{R}^d/\Gamma$.

3. PROOF OF THEOREM 2.1

Let $\mathcal{Q}$ be the set of open parallelepipeds in $\mathbb{R}^d$ related to the lattice Γ, i.e. $\mathcal{Q}$ is the set of all open parallelepipeds Q such that the intersection of the closure of Q and Γ is exactly the set of vertices of Q. Since we can cover I by a countable union of open balls $B(x,r)$ we can without loss of generality assume that $I = B(x_0, r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$.

Fix a parallelepiped $Q_0 \in \mathcal{Q}$. Let

$$\mathcal{B} := \{B(x,r) \bmod \Gamma;\ B(x,r) \subset Q_0, x \in Q_0 \cap \mathbb{Q}^d, r \in \mathbb{Q}^+\}.$$

We will show that, for $B \in \mathcal{B}$, the function

$$F_n(\theta) = \frac{1}{n}\sum_{j=1}^{n} \chi_B\big(f_j(\theta)\big), \qquad n \geqslant 1,$$

fulfills

(1) $$\overline{\lim_{n\to\infty}} F_n(\theta) \leqslant m(B), \qquad \text{for a.e. } \theta \in I.$$

By a standard argument (see e.g. [6]), (1) implies that, for a.e. $\theta \in I$, every weak-$*$ limit point μ_θ of

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{f_j(\theta)}$$

is absolutely continuous with respect to m in $(\mathbb{R}^d/\Gamma) \setminus \partial Q_0$ where the density satisfies $d\mu_\theta/dm \leqslant 1$. Observe that

$$\left|\bigcap_{k\geqslant 1}\{\theta \in I;\, f_j(\theta) \in \partial Q_0 \bmod \Gamma, \text{for some } j \geqslant k\}\right| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$. Hence, for a.e. $\theta \in I$, the probability measure μ_θ gives zero measure to $\partial Q_0 \bmod \Gamma$. It follows that, for a.e. $\theta \in I$, $\mu_\theta = m$ which implies Theorem 2.1.

From now $B \in \mathcal{B}$ is fixed. Let $r_B = \min\{\text{radius}(B), r_0\}$. In order to prove (1) it is sufficient to show that there exists a constant $C > 0$ such that for all $\varepsilon_0 > 0$ and integers $h \geqslant 1$ there is an integer n_{h,ε_0,r_B} growing at most exponentially in h, such that $\int_I F_n(\theta)^h\, d\theta \leqslant C((1+\varepsilon_0)m(B))^h$ for all $n \geqslant n_{h,\varepsilon_0,r_B}$. See 4.2.

Henceforth, fix $h \geqslant 1$.

(2) $$\int_I F_n(\theta)^h\, d\theta = \sum_{1\leqslant j_1,j_2,\ldots,j_h\leqslant n} \frac{1}{n^h}\int_I \chi_B\big(f_{j_1}(\theta)\big)\cdots\chi_B\big(f_{j_h}(\theta)\big)\, d\theta.$$

Let $0 < \kappa < 1$ be the constant in condition (I). The number of h-tuples $(j_1, \ldots, j_h)$ in (2), for which $\min_l j_l < n^\kappa$ or $\min_{k\neq l} |j_k - j_l| < n^\kappa$, is bounded by $2h^2 n^{h-(1-\kappa)}$. The sum over these (exceptional) terms in (2) is therefore bounded by $|I| 2h^2 n^{-(1-\kappa)}$. In the following proposition we treat all the other terms in (2), i.e. the terms related to the h-tuples which are most likely to occur.

Proposition 3.1. *For all $\varepsilon_0 > 0$ and $h \geqslant 1$, there is an integer n_{h,ε_0,r_B} growing at most exponentially in h, such that for all $n \geqslant n_{h,\varepsilon_0,r_B}$ and for all h-tuples $(j_1, \ldots, j_h)$ with $1 \leqslant j_1 < j_2 < \cdots < j_h \leqslant n$, $j_1 \geqslant n^\kappa$ and $j_l - j_{l-1} \geqslant n^\kappa$, $l = 2, \ldots, h$,*

$$\int_I \chi_B\big(f_{j_1}(\theta)\big)\cdots\chi_B\big(f_{j_h}(\theta)\big)\, d\theta \leqslant 2|I|\big((1+\varepsilon_0)m(B)\big)^h.$$

From a probabilistic point of view this proposition tells us that whenever the distances between consecutive j_l's are sufficiently large, the behavior of the $\chi_B(f_{j_l}(.))$'s is comparable to that of independent random variables.

Proposition 3.1 implies

$$\int_I F_n(\theta)^h \, d\theta \leqslant 2|I|\big((1+\varepsilon_0)m(B)\big)^h + |I|2h^2 n^{-(1-\kappa)}$$
$$\leqslant 3|I|\big((1+\varepsilon_0)m(B)\big)^h,$$

for all

$$n \geqslant \max\left\{n_{h,\varepsilon_0,r_B}, \left(\frac{2h^2}{((1+\varepsilon_0)m(B))^h}\right)^{1/(1-\kappa)}\right\}.$$

Since both terms in this lower bound for n grow at most exponentially in h, this concludes the proof of Theorem 2.1. Now we turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $\tilde{B}$ denote the lift of B to $\mathbb{R}^d$. We have

$$\int_I \chi_B\big(f_{j_1}(\theta)\big)\cdots\chi_B\big(f_{j_h}(\theta)\big)\,d\theta = |\{\theta \in I;\ \tilde{f}_{j_1}(\theta) \in \tilde{B}, \ldots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\}|.$$

For $J \subset I$ open and $j \geqslant 1$ we define the partition $\mathcal{P}_j|J$ on J as

$$\mathcal{P}_j|J := \big\{\tilde{f}_j^{-1}\big(Q \cap \tilde{f}_j(J)\big);\ Q \in \mathcal{Q}\big\}.$$

For $j = 0$ we set $\mathcal{P}_0|J = J$ and $\tilde{f}_0(\theta) = \theta$. We give first a sketch of the proof of Proposition 3.1. Note that by the expansion property of the $\tilde{f}_j$'s we have that for large n a typical partition element $\omega \in \mathcal{P}_{j_1}|I$ is mapped by $\tilde{f}_{j_1}$ onto the whole of a parallelepiped $Q \in \mathcal{Q}$, i.e. we can neglect the elements in $\mathcal{P}_{j_1}|I$ adjacent to the boundary of I (the union of these boundary elements is the exceptional set E_0 defined below). By the distortion property of the $\tilde{f}_j$'s we have that roughly speaking only a $|B|/|Q|(= m(B))$ fraction of the element ω is mapped by $\tilde{f}_{j_1}$ onto $\tilde{B} \cap Q$. Considering only the part J of ω which is mapped onto $\tilde{B} \cap Q$ we can now, by using that $j_2 - j_1$ is large, repeat the argument for the elements in the partition $\mathcal{P}_{j_2}|J$. Going on like this we derive Proposition 3.1. In the remaining part we will work this out in more detail.

We say that an element $\omega \in \mathcal{P}_j|J$, $j \geqslant 1$, is an *entire* element if there is a $Q \in \mathcal{Q}$ such that $\tilde{f}_j(\omega) = Q$. Set

$$I_0 = \{\text{entire } \omega \in \mathcal{P}_{j_1}|I\},$$
$$I_l = \{\text{entire } \omega \in \mathcal{P}_{j_{l+1}}|I_{l-1};\ \tilde{f}_{j_l}(\omega) \subset \tilde{B}\},$$

for $1 \leqslant l < h$, and

$$I_h = \{\theta \in I_{h-1};\ \tilde{f}_{j_h}(\theta) \in \tilde{B}\}.$$

We consider the set I_l, $0 \leqslant l < h$, as a set of partition elements in $\mathcal{P}_{j_{l+1}}|I$ as well as an open set in I. Let

$$E_0 = \{\omega \in \mathcal{P}_{j_1}|I;\ \omega \notin I_0\},$$

and, for $1 \leqslant l < h$,

$$E_l = \{\omega \in \mathcal{P}_{j_{l+1}}|I_{l-1};\ \omega \notin I_l,\ \tilde{f}_{j_l}(\omega) \cap \tilde{B} \neq \emptyset\}.$$

Observe that the union of these sets contains (modulo a Lebesgue measure zero set) the set we are interested in, i.e.

$$\{\theta \in I;\ \tilde{f}_{j_1}(\theta) \in \tilde{B}, \dots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\} \overset{\circ}{\subset} I_h \cup \left(\bigcup_{l=0}^{h-1} E_l\right). \tag{3}$$

We state two lemmas. Provided that n is sufficiently large, the first lemma indicates that we can essentially deal with entire elements only, i.e. the E_l's are exceptional sets which can be neglected, and, thus, if $\omega \in \mathcal{P}_{j_{l+1}}|I$, $1 \leqslant l < h$, and $\tilde{f}_{j_l}(\omega) \cap \tilde{B} \neq \emptyset$ then we can assume – without loss of generality – that ω is an entire element. The main ingredient in the proof of this lemma is condition (I). Using condition (II), the second lemma gives a proof of Proposition 3.1 for the 'nice' set I_h.

Lemma 3.2. *For all $\varepsilon > 0$ and $r > 0$, there is an integer $n_{\varepsilon,r}$ growing at most polynomially in $\frac{1}{\varepsilon}$, such that for $n \geqslant n_{\varepsilon,r}$ the following holds. Assume $j = 0$ or $n^\kappa \leqslant j \leqslant n$, and $k \geqslant n^\kappa$. For $B(x,r) \subset \tilde{f}_j(I)$ set $J = \tilde{f}_j^{-1}(B(x,r))$,*

$$J' = \{\text{entire } \omega \in \mathcal{P}_{j+k}|J\},$$

and

$$E_J = \left\{\omega \in \mathcal{P}_{j+k}|I;\ \omega \notin J',\ \tilde{f}_j(\omega) \cap B(x,r) \neq \emptyset\right\}.$$

We have that

$$|E_J| \leqslant \varepsilon |J|.$$

Proof. By condition (I), for $\omega \in \mathcal{P}_{j+k}|I$, $\operatorname{diam}(\tilde{f}_j(\omega)) \leqslant \operatorname{diam}(Q_0)/g(n)$. Hence,

$$|\tilde{f}_j(E_J)| \leqslant \frac{2\operatorname{diam}(Q_0)\operatorname{Vol}_{d-1}(\partial B(x,r))}{g(n)}.$$

Let $j_{1,r}$ be the integer in condition (II). Take $n_{\varepsilon,r} \geqslant (j_{1,r})^{1/\kappa}$ minimal such that for $n \geqslant n_{\varepsilon,r}$,

$$g(n) \geqslant \frac{4\operatorname{diam}(Q_0)\operatorname{Vol}_{d-1}(\partial B(x,r))}{\varepsilon |B(x,r)|}.$$

Recall that, by (I), $g(n)$ is at least polynomially growing in n, hence $n_{\varepsilon,r}$ is at most polynomially growing in $\frac{1}{\varepsilon}$. For $n \geqslant n_{\varepsilon,r}$, we obtain

$$|\tilde{f}_j(E_J)| \leqslant 2^{-1}\varepsilon|B(x,r)|.$$

If $j = 0$ we are done. Otherwise we have $j \geqslant n^{\kappa} \geqslant j_{1,r}$, and it follows by the distortion estimate in condition (II) that $|E_J| \leqslant \varepsilon|J|$. $\square$

Lemma 3.3. *Set* $r_{\Gamma} = \operatorname{diam}(Q_0)/2$ *and let* $j_{\varepsilon_0,r_{\Gamma}}$ *be the integer in condition* (II). *If* $n^{\kappa} \geqslant j_{\varepsilon_0,r_{\Gamma}}$ *then*

$$|I_l| \leqslant (1+\varepsilon_0)m(B)|I_{l-1}|, \quad \textit{for } 1 \leqslant l \leqslant h.$$

Proof. Let $\omega \in I_{l-1}$. By the definition of I_{l-1}, ω is an entire element in $\mathcal{P}_{j_l}|I$. Since $j_l \geqslant n^{\kappa} \geqslant j_{\varepsilon_0,r_{\Gamma}}$ we have by condition (II),

$$|\{\theta \in \omega;\ \tilde{f}_{j_l}(\theta) \in \tilde{B}\}| \leqslant (1+\varepsilon_0)\frac{|\tilde{B} \cap Q_0|}{|Q_0|}|\omega| = (1+\varepsilon_0)m(B)|\omega|.$$

Hence,

$$|\{\theta \in I_{l-1};\ \tilde{f}_{j_l} \in \tilde{B}\}| \leqslant (1+\varepsilon_0)m(B)|I_{l-1}|.$$

Since $I_l \subset \{\theta \in I_{l-1};\ \tilde{f}_{j_l}(\theta) \in \tilde{B}\}$ this concludes the proof. $\square$

Recall that $r_B = \min\{\text{radius}(B), r_0\}$. Let $\varepsilon_1 = ((1+\varepsilon_0)m(B))^h/h$ and n_{ε_1,r_B} the integer in Lemma 3.2, and set

$$n_{h,\varepsilon_0,r_B} = \max\{n_{\varepsilon_1,r_B}, (j_{\varepsilon_0,r_{\Gamma}})^{1/\kappa}\}.$$

Since n_{ε_1,r_B} is at most polynomially growing in $\frac{1}{\varepsilon_1}$, it follows that n_{h,ε_0,r_B} is at most exponentially growing in h. Henceforth, assume $n \geqslant n_{h,\varepsilon_0,r_B}$. Setting $j = 0$, $k = j_1$ and $J = I(= B(x_0, r_0))$ in Lemma 3.2, it follows immediately that $|E_0| \leqslant \varepsilon_1|I|$. Now let $\omega \in I_{l-1}$, $1 \leqslant l < h$. By the definition of I_{l-1}, ω is an entire element in $\mathcal{P}_{j_l}|I$. Set, in Lemma 3.2, $j = j_l$, $k = j_{l+1} - j_l$ and $J = \tilde{f}_{j_l}^{-1}(\tilde{B} \cap \tilde{f}_{j_l}(\omega))$, i.e. J is the part of ω which is mapped by f_{j_l} onto B. Then $J' = I_l \cap \omega$, and we obtain

$$\left|\{\omega' \in \mathcal{P}_{j_{l+1}}|\omega;\ \omega' \notin I_l,\ \tilde{f}_{j_l}(\omega') \cap \tilde{B} \neq \emptyset\}\right| \leqslant \varepsilon_1|J| \leqslant \varepsilon_1|\omega|.$$

Observe that,

$$E_l = \bigcup_{\omega \in I_{l-1}} \{\omega' \in \mathcal{P}_{j_{l+1}}|\omega;\ \omega' \notin I_l,\ \tilde{f}_{j_l}(\omega') \cap \tilde{B} \neq \emptyset\}.$$

Thus $|E_l| \leqslant \varepsilon_1|I_{l-1}| \leqslant \varepsilon_1|I|$. By (3), Lemma 3.3 and the choice of ε_1,

$$\begin{aligned}&|\{\theta \in I;\ \tilde{f}_{j_1}(\theta) \in \tilde{B}, \ldots, \tilde{f}_{j_h}(\theta) \in \tilde{B}\}| \\ &\quad \leqslant |I_h| + \sum_{l=0}^{l-1}|E_l| \leqslant ((1+\varepsilon_0)|B|)^h|I_0| + h\varepsilon_1|I| \leqslant 2|I|((1+\varepsilon_0)|B|)^h,\end{aligned}$$

which concludes the proof of Proposition 3.1. □

4. EXAMPLES

We give two simple examples which can be derived from Theorem 2.1. Note that both examples also can be derived from a combination of Weyl's lemma and Koksma's theorem.

Example 4.1. Let Γ be a lattice in $\mathbb{R}^d$. Fix $\alpha = (\alpha_1, \ldots, \alpha_d) \in (0, \infty)^d$. There is a subset X_α of $(1, \infty)^d$ of full Lebesgue measure, such that the sequence $(\theta_1^{j^{\alpha_1}}, \ldots, \theta_d^{j^{\alpha_d}}) \bmod \Gamma$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for all θ in X_α.

Proof. Let $\tilde{f}_j(\theta) = (\theta_1^{j^{\alpha_1}}, \ldots, \theta_d^{j^{\alpha_d}})$, defined on $I = (\theta_0, \infty)^d$ for some $\theta_0 > 1$. To apply Theorem 2.1, we have to check that (I) and (II) hold for the functions $\tilde{f}_j : I \to \mathbb{R}^d$. Set $\alpha_0 = \min\{1, \alpha_1, \ldots, \alpha_d\}$, and choose $0 < \kappa < 1$ such that $\alpha_0 + \kappa > 1$. Note that for large n,

$$\left(n + n^\kappa\right)^{\alpha_0} - n^{\alpha_0} \geqslant \frac{\alpha_0}{2} \frac{n^{\alpha_0 + \kappa}}{n}.$$

Thus, if $j \leqslant n$ and $k \geqslant n^\kappa$,

$$\frac{|D_\theta \tilde{f}_{j+k}(\theta) v|}{|D_\theta \tilde{f}_j(\theta) v|} \geqslant \theta_0^{(n+n^\kappa)^{\alpha_0} - n^{\alpha_0}},$$

is growing faster than polynomially, for all $v \in \mathbb{R}^d \setminus \{0\}$. This implies (I).

To verify (II), we first observe that any points θ, θ' in $\tilde{f}_j^{-1}(B(x, r) \cap \tilde{f}_j(I))$, are on distance at most $2r / j^{\alpha_0} \theta_0^{j^{\alpha_0} - 1}$. Thus,

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} \prod_{i=1}^{d} \left(\frac{\theta_i}{\theta_i'}\right)^{j^{\alpha_i} - 1} \leqslant \prod_{i=1}^{d} \left(1 + \frac{2r}{j^{\alpha_0} \theta_0^{j^{\alpha_0} - 1}}\right)^{j^{\alpha_i} - 1},$$

where the right-hand side converges uniformly to 1 when j increases. This implies (II). □

Even if the above result tells us that the sequence $(\theta_1^{j^{\alpha_1}}, \ldots, \theta_d^{j^{\alpha_d}}) \bmod \Gamma$, $j \geqslant 1$, is almost surely equidistributed in $\mathbb{R}^d/\Gamma$, it is very hard to determine whether this sequence is equidistributed for some given $\theta \in (1, \infty)^d$. For instance, in one-dimension it is not known whether $(\frac{3}{2})^j$ or e^j modulo 1 are equidistributed in $\mathbb{T}$. However, going in an other direction the next example asserts that j^θ-powers of a fixed $\alpha \in (1, \infty)^d$ are equidistributed, for almost every θ:

Example 4.2. For any $\alpha \in (1, \infty)^d$, there is a full measure subset X_α of $(0, \infty)^d$, such that the sequence $(\alpha_1^{j^{\theta_1}}, \ldots, \alpha_d^{j^{\theta_d}}) \bmod \Gamma$, $j \geqslant 1$, is equidistributed in $\mathbb{R}^d/\Gamma$, for all θ in X_α.

Proof. Let $\tilde{f}_j(\theta) = (\alpha_1^{j^{\theta_1}}, \dots, \alpha_d^{j^{\theta_d}})$, defined on $I = (\theta_0, \infty)^d$ for some $\theta_0 > 0$. Choosing $0 < \kappa < 1$ such that $\theta_0 + \kappa > 1$ condition (I) is verified as in Example 4.1. Let $\alpha_0 = \min\{\alpha_1, \dots, \alpha_d\}$. To establish condition (II), we note that

$$\frac{|D_\theta \tilde{f}_j(\theta)|}{|D_\theta \tilde{f}_j(\theta')|} = \prod_{i=1}^{d} j^{\theta_i - \theta_i'} \alpha_i^{j^{\theta_i} - j^{\theta_i'}},$$

and for large j, $|D_\theta \tilde{f}_j(\theta) v| \geqslant \alpha_0^{j^{\theta_0}}$ for all $v \in \mathbb{R}^d \setminus \{0\}$. For $\theta, \theta' \in \tilde{f}_j^{-1}(B(x,r) \cap \tilde{f}_j(I))$ the distance between θ and θ' is less than $2r\alpha_0^{-j^{\theta_0}}$. Hence, when $j \to \infty$, $j^{\theta_i - \theta_i'}$ converges to 1 and (assuming $\theta_i' < \theta_i$)

$$j^{\theta_i} - j^{\theta_i'} \leqslant j^{\theta_i}(\theta_i - \theta_i') \log j \to 0, \quad j \to \infty.$$

We conclude that condition (II) holds. $\square$

Remark. We have restrained from making very general statements. The methods in this paper can probably be pushed without too much hard labor to nil-manifolds, with conditions (I) and (II) replaced by natural expansion and distortion properties of the cover maps. With some minor restrictions on the maps involved, the methods should also apply to compact locally symmetric spaces of non-compact type.

APPENDIX

For the sake of completeness we add the following fact from measure theory, which we believe has independent interest. Let $I \in \mathcal{B}(\mathbb{R}^d)$, $e_j : I \to [0, \infty)$, $j \geqslant 1$, measurable functions, and set

$$F_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} e_j(\theta).$$

Lemma A.1. *Assume that for all $h \geqslant 1$ there is an integer n_h such that*

$$\int_I F_n(\theta)^h \, d\theta \leqslant C\varepsilon^h,$$

for all $n \geqslant n_h$, where C is some constant independent of h. If the sequence n_h grows at most exponentially in h then it follows that $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant \varepsilon$ for Lebesgue a.e. $\theta \in I$.

Proof. By possibly increasing the n_h's we can assume that $n_h = 2^{hk}$, for some fixed integer k. Let $\delta > 0$ and l, H be integers such that $2^{-l}(2^k - 1) \leqslant \delta$, and $Hk \geqslant l$. Consider the sequence m_i, $i \geqslant 0$, defined as

$$m_i = 2^{(H+[i2^{-l}])k} + \left(i - [i2^{-l}]2^l\right) 2^{(H+[i2^{-l}])k-l}(2^k - 1).$$

This sequence of integers is defined such that, for $h \geqslant H$,

$$(4) \qquad \#\{i; n_h \leqslant m_i < n_{h+1}\} = 2^l,$$

and the distance between two successive m_i's lying in the interval $[n_h, n_{h+1}]$ is constant. Furthermore, one easily verifies that

$$(5) \qquad \frac{m_{i+1}}{m_i} \leqslant 1 + 2^{-l}(2^k - 1) \leqslant 1 + \delta,$$

for $i \geqslant 0$. Using (4), we get

$$\left| \bigcap_{j \geqslant 0} \bigcup_{i \geqslant j} \{F_{m_i} \geqslant (1+\delta)\varepsilon\} \right|$$

$$\leqslant \lim_{j\to\infty} \sum_{i \geqslant j} |\{F_{m_i} \geqslant (1+\delta)\varepsilon\}| \leqslant \lim_{j\to\infty} \sum_{i \geqslant j} \frac{\int_I F_{m_i}(\theta)^{H+[i2^{-l}]}\, d\theta}{((1+\delta)\varepsilon)^{H+[i2^{-l}]}}$$

$$\leqslant \lim_{j\to\infty} C \sum_{i \geqslant j} \left(\frac{\varepsilon}{(1+\delta)\varepsilon} \right)^{H+[i2^{-l}]} \leqslant \lim_{j\to\infty} C 2^l \sum_{h \geqslant j} \frac{1}{(1+\delta)^h} = 0.$$

It follows that $\overline{\lim}_{i\to\infty} F_{m_i}(\theta) \leqslant (1+\delta)\varepsilon$ for all θ in a set I' which has full Lebesgue measure in I. Fix $\theta \in I'$. For sufficiently large i we have $F_{m_i}(\theta) \leqslant (1+\delta)^2\varepsilon$, and using the definition of F_n and inequality (5), we obtain, for $1 \leqslant j < m_{i+1} - m_i$,

$$F_{m_i+j}(\theta) \leqslant \frac{m_{i+1}}{m_i + j} F_{m_{i+1}}(\theta) \leqslant \frac{m_{i+1}}{m_i}(1+\delta)^2\varepsilon \leqslant (1+\delta)^3\varepsilon.$$

It follows that $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant (1+\delta)^3\varepsilon$ for all $\theta \in I'$. Thus, since $\delta > 0$ was arbitrary, $\overline{\lim}_{n\to\infty} F_n(\theta) \leqslant \varepsilon$ for a.e. $\theta \in I$. □

To conclude this appendix we give an example showing that n_h cannot grow arbitrarily fast in h. Let $I = [0, 1]$ and consider the super-exponentially growing sequence $n_h = 3^{2^h}$, $h \geqslant 1$. Set $e_j(a) \equiv 0$ for $j < n_1$ and for $n_h \leqslant j < n_{h+1}$, $h \geqslant 1$, let

$$e_j(a) = \chi_{[\frac{k}{2^{h-1}}, \frac{k+1}{2^{h-1}}]}(a),$$

if $3^{2k}n_h \leqslant j < 3^{2k+1}n_h$, $0 \leqslant k < 2^{h-1}$, and $e_j(a) \equiv 0$ otherwise. It can easily be verified that for each $n_h \leqslant n < n_{h+1}$, the average function $F_n(a)$ is smaller or equal than $1/3$ everywhere except on an interval of length $1/2^{h-1}$ and furthermore, for every $a \in I$, there exists an $n_h \leqslant n < n_{h+1}$ such that $F_n(a) \geqslant 2/3$. It follows that $\overline{\lim}_{n\to\infty} F_n(a) \geqslant 2/3$, for all $a \in I$. On the other hand, for $h \geqslant 1$,

$$\int_I F_n(a)^h\, da \leqslant \left(\frac{1}{3}\right)^h + \frac{1}{2^{h-1}} \leqslant 3\left(\frac{1}{2}\right)^h,$$

if $n \geqslant n_h$.

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